

QUANTUM RING OF SINGULARITY $X^p + XY^q$

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ABSTRACT. In this paper, we will prove that the quantum ring of the quasi-homogeneous polynomial $X^p + XY^q$ ($p \geq 2, q > 1$) with some admissible symmetry group G defined by Fan-Jarvis-Ruan-Witten theory is isomorphic to the Milnor ring of its mirror dual polynomial $X^pY + Y^q$. We will construct a concrete isomorphism between them. The construction is a little bit different in case $(p-1, q) = 1$ and case $(p-1, q) = d > 1$. Some other problems including the correspondence between the pairings of both Frobenius algebras has also been discussed.

1. INTRODUCTION

Let (X, x) be an isolated complete intersection singularity of dimension $N - 1$. This means that X is isomorphic to the fibre $(f^{-1}(0), 0)$ of an analytic map-germ $f : (\mathbb{C}^{N+k-1}, 0) \rightarrow (\mathbb{C}^k, 0)$, and $x \in X$ is an isolated singular point of X . In particular, if $k = 1$, (X, x) is called a hypersurface singularity. The study of the singularity was initiated by H. Whitney, R. Thom and later developed by V. Arnold, K. Saito and many other mathematicians during 60-80 years (See [AGV], [S], [ST], [He]). The classification problem is the central topic in the singularity theory. Many geometrical and topological invariants were introduced to describe the behavior of the singularity, for instance, the Milnor ring, intersection matrix, Gauss-Manin system, periodic map and etc. The singularity theory has tight connection with many fields in mathematics, like differential equations, function theory and symplectic geometry.

Recently, in the papers [FJR1, FJR2, FJR3] the first author and his cooperators has constructed a quantum theory for the hypersurface singularity if the singularity is given by a non-degenerate quasi-homogeneous polynomial W . The start point of their work is Witten's work [Wi2] on the r -spin curves, where Witten wanted to generalize the Witten-Kontsevich theorem [Wi1, Ko] to the moduli problem of r -spin curves (See [AJ], [Ja1, Ja2], [JKV1, JKV2], [PV] for the discussion on the r -spin curves). Unlike in the r -spin case that the Witten equation has only trivial solution, in the general W case, for example D_n and E_7 cases, the Witten equation may have nontrivial solutions which can't be ignored in the construction of the virtual cycle $[\mathcal{W}_{g,k}]^{vir}$. The Witten equation is defined on an orbifold curve and has the following form

$$\bar{\partial}u_i + \frac{\partial \bar{W}}{\partial u_i} = 0,$$

where u_i are sections of appropriate orbifold line bundles.

The Witten equation comes from the study of the Landau-Ginzburg (LG) model in supersymmetric quantum field theory (See [Mar]). It can be viewed as a geometrical realization of the $N = 2$ superconformal algebra. The other known model is the Nonlinear sigma model which corresponds to the Gromov-Witten theory in symplectic geometry. In

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simple case, the LG model is totally determined by a superpotential, which is a quasi-homogeneous polynomial required by supersymmetry. There are two possible ways to get the topological field theories by twisting the LG model. They are called the LG A model and LG B model. The LG B model has been studied extensively in physics and mathematics. The mathematical theory of LG A model (See [GS1, GS2] for the physical explanation) is just the quantum singularity theory constructed by Fan-Jarvis-Ruan. As pointed out in [IV], the more appropriate model is orbifold LG model, which should be "identical" to a Calabi-Yau sigma model by CY/LG correspondence. Actually the state space of the quantum singularity theory is a space of the dual forms of Lefschetz thimbles orbifolding by the admissible symmetric group G of the polynomial W .

Once we determine the state space and obtain the virtual cycle $[\mathcal{W}_{g,k}]^{vir}$, we can build up the quantum invariants for the singularity. For instance, we can define the correlators $\langle \tau_{l_1}(\alpha_{i_1}), \dots, \tau_{l_n}(\alpha_{i_n}) \rangle_g^{W,G}$ for α_{i_j} in the state space $\mathcal{H}_{W,G}$ and the cohomological field theory. All the correlators can be assembled into a generating function

$$\mathcal{D}_{W,G} = \exp\left(\sum_{g \geq 0} \hbar^{2g-2} \mathcal{F}_{g,W,G}\right),$$

where

$$\mathcal{F}_{g,W,G} = \sum_{k \geq 0} \langle \tau_{l_1}(\alpha_{i_1}), \dots, \tau_{l_n}(\alpha_{i_n}) \rangle_g^{W,G} \frac{t_{i_1}^{l_1} \cdots t_{i_n}^{l_n}}{n!}$$

is the genus- g generating function.

So computing out those quantum invariants becomes an important issue to understand the singularity. Because of the Mirror symmetry phenomena between the dual singularities (See [CI] and references there), the quantum ring in the A model of the singularity W should be isomorphic to the Milnor ring in the B model of the dual singularity \tilde{W} (See [IV],[Ka1, Ka2, Ka3]). Furthermore, we have more strong conjecture relating the generating function $\mathcal{D}_{W,G}$ and the formal Givental's generating function. Let us say more about this conjecture.

The genus g Gromov-Witten potential function of one point is

$$\mathcal{F}_g^{pt} := \sum_{k \geq 0} \frac{1}{k!} \sum_{d_1, \dots, d_k} \langle \tau_{d_1} \cdots, \tau_{d_k} \rangle_g t_{d_1} \cdots t_{d_k},$$

where

$$\langle \tau_{d_1} \cdots, \tau_{d_n} \rangle_g = \int_{\mathcal{M}_{g,k}} \psi_1^{d_1} \cdots \psi_k^{d_k}.$$

The Witten-Kontsevich generating function is $\mathcal{D}^{pt} = \exp(\sum_g \hbar^{g-1} \mathcal{F}_g^{pt})$.

Let A be a finite index set having a distinguish element 1. Suppose that the \mathbb{Q} vector space $Vect(A)$ generated by A is attached with a nondegenerate symmetric bilinear form η . The formal genus 0 GW potential is a power series \mathcal{F}_0 in variables $t_{d,l}$, $d \in \mathbb{N}$, $l \in A$,

$$\mathcal{F}_0 = \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{d_1, \dots, d_k \\ l_1, \dots, l_k}} \langle \tau_{d_1, l_1} \cdots, \tau_{d_k, l_k} \rangle_0 t_{d_1, l_1} \cdots t_{d_k, l_k},$$

which satisfies the string equation (SE), the dilaton equation (DE) and the topological recursion equation (TRR).

Let $\mathcal{F}_{pr} = \mathcal{F}|_{\{t_{d_k, l_k} | t_{d_k, l_k} = 0, \text{ for } d_k > 0\}}$ be the primary potential, then \mathcal{F}_{pr} satisfies the WDVV equation and form a Frobenius manifold. \mathcal{F}_0 is called semi-simple of rank μ if $|A| = \mu$ and the algebra structure on $Vect(A)$ is semi-simple for generic $t_{0,l}$. In [Gi], Givental found that

there is a transitive action of the so-called *twisted loop group* on the set of all semi-simple genus 0 GW potential of rank μ . Hence given a semi-simple potential \mathcal{F}_0 of rank μ there is group element R taking k copies $\mathcal{F}_0^{pt} \oplus \cdots \oplus \mathcal{F}_0^{pt}$ to \mathcal{F}_0 .

Using a method to quantize the quadratic functions (see [Gi]), Givental can quantize the group element R to get an element $\hat{R}(\hbar)$ in *Givental's group*. $\hat{R}(\hbar)$ acts on the k copies of the tau-functions $\mathcal{D}^{pt} \oplus \cdots \oplus \mathcal{D}^{pt}$ to get a power series \mathcal{D}_{Giv} in \hbar . \mathcal{D}_{Giv} can be written in the form $\mathcal{D}_{Giv} = \exp(\sum_g \hbar^{g-1} \mathcal{F}_g)$. If \mathcal{D}_{Giv} is required to satisfies a homogeneity condition, then \mathcal{D}_{Giv} is uniquely defined and satisfy the SE, DE, TRR and the Virasoro constraints.

If given a genus 0 GW potential of a projective manifold which is semi-simple, Givental conjectured that the total GW potential is the same to \mathcal{D}_{Giv} constructed from the genus 0 GW potential. We have the similar question in the quantum singularity theory. Let $\mathcal{D}_{W,G}$ be the tau-function in our Landau-Ginzburg A model and $\mathcal{D}_{0,W,G}$ be the genus 0 tau-function. If the Frobenius manifold induced by $\mathcal{D}_{0,W,G}$ is semi-simple, then we can get the formal tau-function $\mathcal{D}_{Giv,W,G}$.

Conjecture 1.1. $\mathcal{D}_{W,G} = \mathcal{D}_{Giv,W,G}$

This should be true by Teleman's theorem [Te] if we can show that $\mathcal{D}_{0,W,G}$ is semi-simple. To prove the semi-simple property, it is natural to show the Frobenius manifold associated to the singularity W/G in the A model is isomorphic to the Saito's Frobenius manifold of the dual singularity \check{W} in the B model, which is easy proved to be semi-simple. If the symmetry group G is chosen suitable, we should have the following problem

Conjecture 1.2. $\mathcal{D}_{W,G}$ is identical to $\mathcal{D}_{Giv,\check{W}}$ under some Mirror transformation.

Conversely, since the construction of the quantum theory depends on the choice of the admissible subgroup G such that $\langle J \rangle \leq G \leq G_W$ (See the definitions in Section 2), we can't expect a mirror correspondence from the LG A model of the dual singularity \check{W} with the trivial symmetry group to the LG B model of the singularity W . A further discussion will be appeared in [Kr].

In [FJR2], the authors has calculated the quantum ring structure of the ADE singularities. Moreover by computing the basic 4 point correlators, using the WDVV equation and the reconstruction theorems, the authors has proved the above conjecture and thus proved the generalized Witten conjecture for DE cases via the conclusions in [GM].

ADE singularities are simple singularities according to Arnold's classification and has very special properties. For instance ADE singularities are self-Mirror which has been shown in [FJR2].

To prove the Conjecture 1.2 for general singularity, one has to compare the Frobenius manifolds in both sides. Even in the B side it is difficult to calculate the primary potential of Saito's Frobenius manifold associated to a singularity other than ADE singularities. One can consider the singularities with modality no less than 1. The computation of the quantum ring of those singularities has recently been done (See [Kr],[Pr]). On the other hand, M. Noumi [No] has considered the following type singularities:

- (i) $x_1^{p_1} + x_2^{p_2} + \cdots + x_N^{p_N}$
- (ii) $x_1^{p_1} + x_1 x_2^{p_2} + x_3^{p_3} + \cdots + x_N^{p_N}$.

He has considered the Gauss-Manin system associated to the above singularity. An important fact is that the flat coordinates on the Frobenius manifold are the polynomials of the deformed coordinates appeared in the miniversal deformation, and meanwhile the formula of primary potential was given in [NY].

Since the Frobenius structure of the above singularities in either side is the tensor product of the Frobenius structures of the A_r singularity and the singularity $x^p + xy^q$, it is natural

for us only to compute the primary potential functions of the singularity $x^p + xy^q$ in A model and then compare it with Noumi-Yamada's computation in B model. By WDVV equation, one may show that the primary potential depends on the 2, 3 point correlators and some basic 4 correlators. We need only compare the 2, 3 point correlators and some 4 correlators in both sides. The computation of the quantum invariants of x^p and $x^p + xy^q$ is important, since we can take the direct sum of those singularities to form a Calabi-Yau singularity (whose central charge is positive integer). Once we know the quantum invariants of the Calabi-Yau singularity, then by CY/LG correspondence it is hopeful to get the Gromov-Witten invariants of the Calabi-Yau hypersurface defined by the CY singularity. Actually a computation has been done in [CR] for quintic three-fold.

In this paper, we will calculate the quantum ring structure of the singularity $x^p + xy^q$, $p \geq 2, q > 1$ and construct the explicit isomorphism to the Milnor ring of the dual singularity $x^p y + y^q$ in Berglund-Hübsch sense (see [BH]). In a subsequent paper, we will do the difficult computation of the basic 4 point correlators and build the isomorphism between two Frobenius manifolds. This paper is arranged as follows. Section 2 gives a simple description of the Fan-Jarvis-Ruan theory and list some useful axioms. In Section 3, we will discuss the singularity $x^p + xy^q$ in case that $(p-1, q) = 1$. In section 4, we will treat the case that $(p-1, q) = d > 1$. In addition, in section 2, we also write down the equivalence of the pairing of the dual forms of Lefschetz thimbles and the residue pairing in the Milnor ring. This is just the Mirror symmetry between the 2 point functions. Though this fact was mentioned in [FJR2] and appeared in physical literature (see [Ce]), it is seldom known by mathematicians and is deserved to be written down.

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2. THE FAN-JARVIS-RUAN-WITTEN THEORY

The classical singularity theory. A polynomial $W : \mathbb{C}^N \rightarrow \mathbb{C}$ is called quasi-homogeneous if there are positive integers d, n_1, \dots, n_N such that $W(\lambda^{n_1} x_1, \dots, \lambda^{n_N} x_N) = \lambda^d W(x_1, \dots, x_N)$. We define the *weight* (or *charge*), of x_i to be $q_i := \frac{n_i}{d}$. We say W is *nondegenerate* if (1) the choices of weights q_i are unique, and (2) W has a singularity only at zero. There are many examples of non-degenerate quasi-homogeneous singularities, including all the nondegenerate homogeneous polynomials and the famous *ADE*-examples:

Example 2.1.

$$\begin{aligned} A_n: & W = x^{n+1}, \quad n \geq 1; \\ D_n: & W = x^{n-1} + xy^2, \quad n \geq 4; \\ E_6: & W = x^3 + y^4; \\ E_7: & W = x^3 + xy^3; \\ E_8: & W = x^3 + y^5; \end{aligned}$$

A classical invariant of the singularity is the *local algebra*, also known as the *Milnor ring* or *Chiral ring* in physics:

$$\mathcal{Q}_W := \mathbb{C}[x_1, \dots, x_N] / \text{Jac}(W), \quad (1)$$

where $\text{Jac}(W)$ is the Jacobian ideal, generated by partial derivatives

$$\text{Jac}(W) := \left(\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_N} \right).$$

The degree of the monomial makes the local algebra become a graded algebra. There is a unique highest-degree element $\det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$ with degree

$$\hat{c}_W = \sum_i (1 - 2q_i). \quad (2)$$

which is called the *central charge* of W .

The dimension of the local algebra is called the *Milnor number* and is given by the formula

$$\mu = \prod_i \left(\frac{1}{q_i} - 1 \right).$$

Let S be a small ball centered at the origin of \mathbb{C}^μ and consider the miniversal deformation $F(x, t)$ of W such that $F(x, 0) = W$. We have the Milnor fibration $F : \mathbb{C}^N \times S \rightarrow \mathbb{C} \times S$ given by $(x, t) \rightarrow (F(x, t), t)$. Assume that the critical value of F are in $\mathbb{C}_\delta \times S$, where $\mathbb{C}_\delta := \{z \in \mathbb{C} : \|z\| < \delta\}$. Let $z_0 \in \partial\mathbb{C}_\delta$, then $F^{-1}(z_0, t) \rightarrow t \in S$ is a fiber bundle. This induces the homology bundle $H_{N-1}(F^{-1}(z_0, t), \mathbb{Z}) \rightarrow S$. For a generic t , $F(x, t)$ is a holomorphic Morse function. A distinguished basis of $H_{N-1}(F^{-1}(z_0, t), \mathbb{Z})$ can be constructed from a system of paths connecting z_0 to the critical values. A system of paths $l_i : [0, 1] \rightarrow \mathbb{C}_\delta$ connecting z_0 to critical values z_i is called *distinguished* if

- (1) l_i has no self intersection;
- (2) l_i, l_j has no intersection except $l_i(0) = l_j(0) = z_0$;
- (3) the paths l_1, \dots, l_μ are numbered in the same order in which they enter the point z_0 , counter-clockwise.

For each l_i , we can associate a homology class $\delta_i \in H_{N-1}(F^{-1}(z_0, t), \mathbb{Z})$ as a vanishing cycle along l_i . More precisely, the neighborhood of the critical point of z_i contains a local vanishing cycle. Then δ_i is obtained by transporting the local vanishing cycle to z_0 using the homotopy lifting property. The cycle δ_i is unique up to the homotopy of l_i as long as the homotopy does not pass another critical value. Now $\delta_1, \dots, \delta_\mu$ defines a distinguished basis of $H_{N-1}(F^{-1}(z_0, t), \mathbb{Z})$. The different choice of the distinguished system of paths gives different distinguished basis. The transformation relation between two basis is described by the Picard-Lefschetz transformation. The intersection matrix $(\delta_i \circ \delta_j)$ is an invariant of the singularity and is used to classify the singularity. Except the vanishing cycles, another closely related objects are *Lefschetz thimbles*, which are the generators of the relative homology classes $H_N(\mathbb{C}^N, F^{-1}(z_0, t), \mathbb{Z})$. The boundary homomorphism ∂ gives an isomorphism $\partial : H_N(\mathbb{C}^N, F^{-1}(z_0, t), \mathbb{Z}) \rightarrow H_{N-1}(F^{-1}(z_0, t), \mathbb{Z})$. Geometrically, a Lefschetz thimble Δ_i is the union of the vanishing cycles along the path l_i and we have $\partial\Delta_i = \delta_i$.

We can let the radius δ of \mathbb{C}_δ goes to ∞ and take $z_0 = -\infty$. The relative homology class becomes $H_N(\mathbb{C}^N, (ReF)^{-1}((-\infty, -M), t), \mathbb{Z})$ for large $M > 0$. We simply write $(ReF)^{-1}((-\infty, -M), t)$ as $F_t^{-\infty}$ and $(ReF)^{-1}((M, +\infty), t)$ as $F_t^{+\infty}$. Now the Lefschetz thimble Δ_i in $H_N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{Z})$ is canonically determined by the horizontal path from the critical value to $-\infty$.

Unlike the intersection matrix of the vanishing cycles, there is a non-degenerate intersection pairing

$$I : H_N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{Z}) \otimes H_N(\mathbb{C}^N, F_t^{+\infty}, \mathbb{Z}) \rightarrow \mathbb{Z}. \quad (3)$$

This pairing is given by the intersection of the stable manifold and the unstable manifold of the critical point and is preserved by the parallel transportation via the Gauss-Manin connection. Naturally we have the dual pairing (See [FJR2]):

$$\eta : H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C}) \otimes H^N(\mathbb{C}^N, F_t^{+\infty}, \mathbb{C}) \rightarrow \mathbb{C}$$

The Quantum invariants of the singularity. Let $G_W := \text{Aut}(W)$ be the *maximal diagonal symmetry group* of W consisting of the diagonal matrix γ such that $W(\gamma x) = W(x)$. G_W always contains the subgroup $\langle J \rangle$, where $J = \text{diag}(e^{2\pi i q_1}, \dots, e^{2\pi i q_N})$ is the *exponential grading element*. We can take any subgroup G such that $\langle J \rangle \leq G \leq G_W$. Using the group G , we can orbifold the space of Lefschetz thimbles. For any $\gamma \in G$, let \mathbb{C}_γ^N be the set of fixed points of γ , let N_γ denotes its complex dimension, and let $W_\gamma := W|_{\mathbb{C}_\gamma^N}$ be the quasi-homogeneous singularity restricted to the fixed point locus of γ . According to the Lemma 3.2.1 in [FJR2], 0 is the only critical point of W_γ and G is the subgroup of $\text{Aut}(W_\gamma)$.

Definition 2.2. The γ -twisted sector \mathcal{H}_γ of the state space is defined as the G -invariant part of the middle-dimensional relative cohomology for W_γ ; that is,

$$\mathcal{H}_\gamma := H^{N_\gamma}(\mathbb{C}_\gamma^{N_\gamma}, W_\gamma^\infty, \mathbb{C})^G. \quad (4)$$

Definition 2.3. Suppose that $\gamma = (e^{2\pi i \Theta_1^\gamma}, \dots, e^{2\pi i \Theta_N^\gamma}) \in G$ for rational numbers $0 \leq \Theta_i^\gamma < 1$. The *degree shifting number* is $\iota_\gamma := \sum_i (\Theta_i^\gamma - q_i)$ and for a class $\alpha \in \mathcal{H}_\gamma$, we have the definition of the degree

$$\deg_{\mathbb{C}}(\alpha) := \deg_W(\alpha)/2 := \deg(\alpha)/2 + \iota_\gamma.$$

The following proposition was proved in Proposition 3.2.4 in [FJR2].

Proposition 2.4. For any $\gamma \in G_W$ we have the equalities

$$\begin{aligned} \iota_\gamma + \iota_{\gamma^{-1}} &= \hat{c}_W - N_\gamma \\ \deg_{\mathbb{C}}(\alpha) + \deg_{\mathbb{C}}(\beta) &= \hat{c}_W \end{aligned} \quad (5)$$

for any $\alpha \in \mathcal{H}_\gamma$ and $\beta \in \mathcal{H}_{\gamma^{-1}}$.

Definition 2.5. The *state space* of the singularity W/G is defined as

$$\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma.$$

The pairing in $\mathcal{H}_{W,G}$ is defined as the direct sum of the pairings

$$\langle \cdot, \cdot \rangle_\gamma : \mathcal{H}_\gamma \otimes \mathcal{H}_{\gamma^{-1}} \rightarrow \mathbb{C}$$

, where $\langle \cdot, \cdot \rangle_\gamma$ is just the pairing $\eta(\cdot, \cdot)$ of the singularity W_γ .

The quantum invariants of the singularity W/G are defined via the construction of the virtual fundamental cycle $[\mathcal{W}_{g,k}(\gamma)]^{\text{vir}}$ (or $[\mathcal{W}(\Gamma)]^{\text{vir}}$). Let us briefly describe the properties of these virtual fundamental cycle and some axioms related to our computation in this paper. We only consider the case $G = G_W$ or $\langle J \rangle$.

Given a non-degenerate quasi-homogeneous polynomial W , we can define the W -structure on an orbicurve with genus g and k marked points. Roughly speaking, the W structure on a orbicurve \mathcal{C} is a choice of N orbifold line bundles $\mathcal{L}_1, \dots, \mathcal{L}_N$ satisfying some relations defined by the polynomial W . If a W -structure exists on an orbicurve \mathcal{C} , then there must have

$$\deg(|\mathcal{L}_j|) = \left(q_j(2g - 2 + k) - \sum_{l=1}^k \Theta_l^{\gamma_j} \right) \in \mathbb{Z}. \quad (6)$$

Here $\gamma_l = (e^{2\pi i \Theta_1^{\gamma_l}}, \dots, e^{2\pi i \Theta_N^{\gamma_l}}) \in G_W$ gives the orbifold action of the line bundles \mathcal{L}_i at the marked point z_l and $|\mathcal{L}_j|$ is the resolved line bundles on the coarse curve of \mathcal{C} . (see [FJR2] for the detail definition of these structures).

The orbicurve with W -structure is called W -orbicurve. The stack of stable W -orbicurves forms the moduli space $\mathcal{W}_{g,k}$. For any choice $\gamma := (\gamma_1, \dots, \gamma_k) \in G_W^k$ we define $\mathcal{W}_{g,k}(\gamma) \subseteq \mathcal{W}_{g,k}$ to be the open and closed substack with orbifold decoration γ . We call γ the *type* of any W -orbicurve in $\mathcal{W}_{g,k}(\gamma)$. $\mathcal{W}_{g,k}(\gamma)$ is not empty iff the condition (6) holds. Forgetting the W -structure and the orbifold structure gives a morphism

$$st : \mathcal{W}_{g,k} \rightarrow \overline{\mathcal{M}}_{g,k}.$$

The morphism st plays a role similar to that played by the stabilization morphism of stable maps in symplectic geometry. The following theorem is proved in Theorem 2.2.6 of [FJR2].

Theorem 2.6. *For any nondegenerate, quasi-homogeneous polynomial W , the stack $\mathcal{W}_{g,k}$ is a smooth, compact orbifold (Deligne-Mumford stack) with projective coarse moduli. In particular, the morphism $st : \mathcal{W}_{g,k} \rightarrow \overline{\mathcal{M}}_{g,k}$ is flat, proper and quasi-finite (but not representable).*

Moreover, one can consider the decorated dual graph Γ of a stable W -curve and obtain the moduli space $\mathcal{W}_{g,k}(\Gamma)$, which is a closed substack of $\mathcal{W}_{g,k}(\gamma)$.

Let $T(\Gamma)$ be the set of tails of the decorated graph Γ and attach an element $\gamma_\tau \in G_W$ to each tail τ . The virtual cycle $[\mathcal{W}(\Gamma)]^{vir}$ was constructed in these papers [FJR2, FJR3] by studying the Witten equation and its moduli problem. It was proved that the virtual cycle $[\mathcal{W}(\Gamma)]^{vir}$ satisfies a series of axioms analogous to the Kontsevich-Manin axiom system in symplectic geometry. We only list those axioms that we mainly used in this paper.

Set

$$D := - \sum_{i=1}^N \text{index}(\mathcal{L}_i) = \hat{c}_W(g-1) + \sum_{j=1}^k \iota_{\gamma_j}. \quad (7)$$

Theorem 2.7.

(1) **Dimension:** *The cycle $[\mathcal{W}(\Gamma)]^{vir}$ has degree*

$$6g - 6 + 2k - 2D = 2 \left((\hat{c} - 3)(1 - g) + k - \sum_{\tau \in T(\Gamma)} \iota_{\gamma_\tau} \right). \quad (8)$$

So the cycle lies in $H_r(\mathcal{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}_{\gamma_\tau}^N, W_{\gamma_\tau}^\infty, \mathbb{Q})$, where

$$r := 6g - 6 + 2k - 2D - \sum_{\tau \in T(\Gamma)} N_{\gamma_\tau} = 2 \left((\hat{c} - 3)(1 - g) + k - \sum_{\tau \in T(\Gamma)} \iota(\gamma_\tau) - \sum_{\tau \in T(\Gamma)} \frac{N_{\gamma_\tau}}{2} \right).$$

(2) **Degenerating connected graphs:** *Let Γ be a connected, genus- g , stable, decorated W -graph. The cycles $[\mathcal{W}(\Gamma)]^{vir}$ and $[\mathcal{W}_{g,k}(\gamma)]^{vir}$ are related by*

$$[\mathcal{W}(\Gamma)]^{vir} = \tilde{i}^* [\mathcal{W}_{g,k}(\gamma)]^{vir}, \quad (9)$$

where $\tilde{i} : \mathcal{W}(\Gamma) \rightarrow \mathcal{W}_{g,k}(\gamma)$ is the canonical inclusion map.

(3) **Concavity:**

Suppose that all tails of Γ are Neveu-Schwarz. If $\pi_ \left(\bigoplus_{i=1}^t \mathcal{L}_i \right) = 0$, then the virtual cycle is given by capping the top Chern class of the dual $\left(R^1 \pi_* \left(\bigoplus_{i=1}^t \mathcal{L}_i \right) \right)^*$*

of the pushforward with the usual fundamental cycle of the moduli space:

$$\begin{aligned} [\mathcal{W}(\Gamma)]^{vir} &= c_{top} \left(\left(R^1 \pi_* \bigoplus_{i=1}^t \mathcal{L}_i \right)^* \right) \cap [\mathcal{W}(\Gamma)] \\ &= (-1)^D c_D \left(R^1 \pi_* \bigoplus_{i=1}^t \mathcal{L}_i \right) \cap [\mathcal{W}(\Gamma)]. \end{aligned} \quad (10)$$

- (4) **Index zero:** Suppose that $\dim(\mathcal{W}(\Gamma)) = 0$ and all the decorations on tails are Neveu-Schwarz.

If the pushforwards $\pi_* \left(\bigoplus \mathcal{L}_i \right)$ and $R^1 \pi_* \left(\bigoplus \mathcal{L}_i \right)$ are both vector bundles of the same rank, then the virtual cycle is just the degree $\deg(\mathcal{D})$ of the Witten map times the fundamental cycle:

$$[\mathcal{W}(\Gamma)]^{vir} = \deg(\mathcal{D}) [\mathcal{W}(\Gamma)],$$

- (5) **Composition law:** Given any genus g decorated stable W -graph Γ with k tails, and given any edge e of Γ , let $\widehat{\Gamma}$ denote the graph obtained by “cutting” the edge e and replacing it with two unjoined tails τ_+ and τ_- decorated with γ_+ and γ_- , respectively.

The fiber product

$$F := \mathcal{W}(\widehat{\Gamma}) \times_{\mathcal{W}(\Gamma)} \mathcal{W}(\Gamma)$$

has morphisms

$$\mathcal{W}(\widehat{\Gamma}) \xleftarrow{q} F \xrightarrow{pr_2} \mathcal{W}(\Gamma).$$

We have

$$\left\langle [\mathcal{W}(\widehat{\Gamma})]^{vir} \right\rangle_{\pm} = \frac{1}{\deg(q)} q_* pr_2^* ([\mathcal{W}(\Gamma)]^{vir}), \quad (11)$$

where $\langle \rangle_{\pm}$ is the map from

$$H_*(\mathcal{W}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_{\tau}}}(\mathbb{C}_{\gamma_{\tau}}^N, W_{\gamma_{\tau}}^{\infty}, \mathbb{Q}) \otimes H_{N_{\gamma_+}}(\mathbb{C}_{\gamma_+}^N, W_{\gamma_+}^{\infty}, \mathbb{Q}) \otimes H_{N_{\gamma_-}}(\mathbb{C}_{\gamma_-}^N, W_{\gamma_-}^{\infty}, \mathbb{Q})$$

to

$$H_*(\mathcal{W}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_{\tau}}}(\mathbb{C}_{\gamma_{\tau}}^N, W_{\gamma_{\tau}}^{\infty}, \mathbb{Q})$$

obtained by contracting the last two factors via the pairing

$$\langle , \rangle : H_{N_{\gamma_+}}(\mathbb{C}_{\gamma_+}^N, W_{\gamma_+}^{\infty}, \mathbb{Q}) \otimes H_{N_{\gamma_-}}(\mathbb{C}_{\gamma_-}^N, W_{\gamma_-}^{\infty}, \mathbb{Q}) \rightarrow \mathbb{Q}$$

Cohomological field theory. For any homogeneous elements $\alpha := (\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \mathcal{H}_{\gamma_i}$, the map $\Lambda_{g,k}^W \in \text{Hom}(\mathcal{H}_W^{\otimes k}, H^*(\overline{\mathcal{M}}_{g,k}))$ is defined by

$$\Lambda_{g,k}^W(\alpha) := \frac{|G|^g}{\deg(st)} PD \, st_* \left([\mathcal{W}_{g,k}(W, \gamma)]^{vir} \cap \prod_{i=1}^k \alpha_i \right), \quad (12)$$

and then extend linearly to general elements of $\mathcal{H}_W^{\otimes k}$. Here, PD is the Poincare duality map.

The following results were showed in [FJR2]:

Theorem 2.8. *The collection $(\mathcal{H}_W, \langle \cdot, \cdot \rangle^W, \{\Lambda_{g,k}^W\}, \mathbf{e}_1)$ is a cohomological field theory with flat identity.*

Moreover, if W_1 and W_2 are two singularities in distinct variables, then the cohomological field theory arising from $W_1 + W_2$ is the tensor product of the cohomological field theories arising from W_1 and W_2 :

$$(\mathcal{H}_{W_1+W_2}, \{\Lambda_{g,k}^{W_1+W_2}\}) = (\mathcal{H}_{W_1} \otimes \mathcal{H}_{W_2}, \{\Lambda_{g,k}^{W_1} \otimes \Lambda_{g,k}^{W_2}\}).$$

Corollary 2.9. *The genus-zero theory defines a Frobenius manifold.*

The quantum invariants of the singularity W/G_W consists of the correlators defined as below:

Definition 2.10. Define correlators

$$\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_k}(\alpha_k) \rangle_g^W := \int_{[\overline{\mathcal{M}}_{g,k}]} \Lambda_{g,k}^W(\alpha_1, \dots, \alpha_k) \prod_{i=1}^k \psi_i^{l_i},$$

where ψ_i are the canonical classes in the tautological ring of $\overline{\mathcal{M}}_{g,k}$.

For an admissible group G such that $\langle J \rangle \leq G \leq G_W$, we can also define the virtual cycle $[\mathcal{W}_{g,k,G}^{vir}]$, the morphism $\Lambda_{g,k}^{W,G}$ and the correlators $\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_k}(\alpha_k) \rangle_g^{W,G}$. See the discussion in [FJR2].

Quantum ring (Quantum cohomology group) of the singularity. The simplest quantum structure of a singularity is the Frobenius algebra consisting of the state space, the metric and the quantum multiplication \star . The multiplication is given by the genus 0 3-point correlators:

$$\langle \alpha \star \beta, \gamma \rangle = \langle \tau_0(\alpha), \tau_0(\beta), \tau_0(\gamma) \rangle_0^{W,G}. \quad (13)$$

To show the mirror symmetry between the LG A model of the quasi-homogenous singularity W and the LG B model of the dual singularity \check{W} , the first step is to identify the corresponding Frobenius algebra structures and the second step is to compare the Frobenius manifold structure. When the Frobenius manifold structures are identical, it is hopeful to construct the mirror map between the A model theory: Fan-Jarvis-Ruan-Witten theory and the B model theory: Saito-Givental's theory.

Let us write down the explicit correspondence of the metric in A model and the metric in B model. The identification was mentioned in [FJR2] but not explicitly written down.

In LG B model, the Frobenius algebra is the Milnor ring \mathcal{Q}_W with the residue pairing and the multiplication of the monomials. For $f, g \in \mathcal{Q}_W$, the residue pairing is non-degenerate and is defined by

$$\langle f, g \rangle = \text{Res}_{x=0} \frac{fg dx_1 \wedge \dots \wedge dx_N}{\frac{\partial W}{\partial x_1} \dots \frac{\partial W}{\partial x_N}}.$$

Let $\phi_i, i = 0, 1, \dots, \mu - 1$ be the basis of \mathcal{Q}_W . We can also consider the miniversal deformation $F(x, t) = F_t(x) := W + t_0\phi_0 + \dots + t_{\mu-1}\phi_{\mu-1}$ and the deformed Milnor ring with residue pairing $\langle \cdot, \cdot \rangle_t = \text{Res}_t$ at the point $t \in \mathbb{C}^\mu$.

In the A model side, the intersection pairing I of the Lefschetz thimbles has the dual pairing

$$\eta : H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C}) \otimes H^N(\mathbb{C}^N, F_t^{\infty}, \mathbb{C}) \rightarrow \mathbb{C}$$

. The relative cohomology groups $H^N(\mathbb{C}^N, F_t^{\pm\infty}, \mathbb{C})$ and the pairing can be described in forms and the integration of forms on \mathbb{C}^N .

Let $\bar{\partial}_{F_t} := \bar{\partial} + dF_t \wedge$, $\partial_{F_t} := \partial + d\bar{F}_t \wedge$, and $A^{p,q}$ be the set of (p, q) -forms on \mathbb{C}^N . Then one can show that the spectral sequence of the double complex $(A^{*,*}, \bar{\partial}, dF_t \wedge)$ converges to the homology group of $(A^{*,*}, \bar{\partial}_{F_t})$, which is also isomorphic to the Koszul complex (Ω^*, dF_t) . We obtain the isomorphisms

$$H_{\bar{\partial}_{F_t}}^N \simeq \Omega^N / dF_t \wedge d\Omega^{N-1} \simeq \mathcal{L}_{F_t}. \quad (14)$$

Let $\{1 =: \phi_0(x), \dots, \phi_{\mu-1}(x)\}$ be a \mathbb{C} -basis in the Milnor ring \mathcal{L}_{F_t} and $\omega = dx^1 \cdots dx^N$ be the holomorphic volume form in \mathbb{C}^N . Then the above isomorphisms can be given canonically:

$$\phi_i(x)\omega + \bar{\partial}_{F_t}\eta_i \longleftrightarrow \phi_i(x)\omega \longleftrightarrow \phi_i(x).$$

One can also study the cohomology group $H_{\bar{\partial}_{F_t}}^N$ which is isomorphic to $H_{\bar{\partial}_{F_t}}^N$ by a usual conjugation.

It is known that we can choose a family of primitive n -forms $\{\omega_i\}$ ($\{\bar{\omega}_i\}$) generating $H_{\bar{\partial}_{F_t}}^N$ ($H_{\partial_{F_t}}^N$). Such forms are called vacuum wave forms in physical literature (see [Ce]). Note that $\bar{\partial}_{F_t}\omega_i = \partial_{F_t}\omega_i = 0$. The closed forms $\{e^{F_t+\bar{F}_t}\omega_i\}$ ($\{e^{-(F_t+\bar{F}_t)}\bar{\omega}_i\}$) form a basis of $H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C})$ ($H^N(\mathbb{C}^N, F_t^{\infty}, \mathbb{C})$). Let $\{\Delta_a^-, a = 1, \dots, \mu\}$ be a basis of $H_N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C})$ and $\{\Delta_b^+, b = 1, \dots, \mu\}$ be a basis of $H_N(\mathbb{C}^N, F_t^{\infty}, \mathbb{C})$. Define

$$\Pi_a^i = (-1)^{N/2}(2\pi)^{-N/2} \int_{\Delta_a^-} e^{F_t+\bar{F}_t}\omega_i, \quad \tilde{\Pi}_b^j = (-1)^{N/2}(2\pi)^{-N/2} \int_{\Delta_b^+} e^{-(F_t+\bar{F}_t)}\bar{\omega}_j.$$

Now the poincare dual $PD(\Delta_b^+)$ lies in $H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C})$, and we assume that $PD(\Delta_b^+) = \sum_i c_i e^{F_t+\bar{F}_t}\omega_i$. Define $\eta_{ij} = (-1)^N(2\pi)^{-N} \int_{\mathbb{C}^N} \omega_i \wedge * \bar{\omega}_j$. By the relation

$$\int_{\mathbb{C}^N} PD(\Delta_b^+) \wedge e^{-(F_t+\bar{F}_t)}\bar{\omega}_j = \sum_i c_i \int_{\mathbb{C}^N} \omega_i \wedge * \bar{\omega}_j = (-1)^N(2\pi)^N \sum_i c_i \eta_{ij},$$

we have

$$c_i = (-1)^N(2\pi)^{-N} \sum_j \eta^{ij} \int_{\Delta_b^+} e^{-(F_t+\bar{F}_t)}\bar{\omega}_j.$$

So we can compute the intersection number

$$\begin{aligned} I_{a-b^+} &= \#(\Delta_a^- \cap \Delta_b^+) = \int_{\Delta_a^-} PD(\Delta_b^+) \\ &= (-1)^N(2\pi)^{-N} \sum_{i,j} \int_{\Delta_a^-} e^{F_t+\bar{F}_t}\omega_i \eta^{ij} \int_{\Delta_b^+} e^{-(F_t+\bar{F}_t)}\bar{\omega}_j \\ &= \sum_{ij} \Pi_a^i \eta^{ij} \tilde{\Pi}_b^j. \end{aligned}$$

Remark 2.11. The vacuum wave forms ω_i can be chosen in the form

$$\omega_i = \phi_i dx^1 \cdots dx^N + \bar{\partial}_{F_t}\eta_i \quad (15)$$

The following result can be found in [Ce].

Proposition 2.12. Let ω_i have the representation (15) and $\eta_{ij} = (-1)^N(2\pi)^{-N} \int_{\mathbb{C}^N} \omega_i \wedge * \bar{\omega}_j$, then

$$\eta_{ij} = J(\phi_i dx^1 \wedge \cdots \wedge dx^N, \phi_j dx^1 \wedge \cdots \wedge dx^N), \quad (16)$$

where

$$J(\phi_i dx^1 \wedge \cdots \wedge dx^N, \phi_j dx^1 \wedge \cdots \wedge dx^N) := \text{Res}_{x=0} \left(\frac{\phi_i \phi_j dx^1 \wedge \cdots \wedge dx^N}{\frac{\partial F_t}{\partial x^1} \cdots \frac{\partial F_t}{\partial x^N}} \right) \quad (17)$$

is the pairing in $\Omega^N/dF_t \wedge d\Omega^{N-1}$.

This proposition shows that to compute the pairing $\eta : H^N(\mathbb{C}^N, F_t^{-\infty}, \mathbb{C}) \otimes H^N(\mathbb{C}^N, F_t^\infty, \mathbb{C}) \rightarrow \mathbb{C}$ we need only compute the residue pairing of the corresponding polynomials. Since the residue pairing is well-defined at $t = 0$, we can naturally extend the pairing η at $t \neq 0$ to $t = 0$ by identifying it with the residue pairing. The identification will be preserved if we consider the G -invariant theory.

3. QUANTUM RING OF $X^p + XY^q$; $(p-1, q) = 1$

3.1. Basic calculation. Consider the singularity $W = x^p + xy^q$ with the constraint $(p-1, q) = 1$, $p \geq 2, q > 1$. In this case, the group $G = \langle J \rangle \cong \mathbb{Z}/(pq)\mathbb{Z}$. Let $\xi = \exp(\frac{2\pi i}{pq})$, then J acts on $\mathcal{L}_W \omega$ by (ξ^p, ξ^{p-1}) .

We have the computation:

$$\begin{aligned} q_x &= \frac{1}{p}, & q_y &= \frac{p-1}{pq}, & \hat{c}_W &= \frac{2(p-1)(q-1)}{pq} \\ \Theta_x^J &= \frac{1}{p}, & \Theta_y^J &= \frac{p-1}{pq}. \end{aligned}$$

It is easy to obtain the state space

$$\mathcal{H}_{W,G} = \langle y^{q-1} \mathbf{e}_0, \mathbf{e}_k | k \in \Lambda \rangle,$$

where $\Lambda = \{i \mid 1 \leq i \leq pq-1, p \nmid i\}$, $\mathbf{e}_0 := dx \wedge dy \in H^{mid}(\mathbb{C}_{J^0}^N, W_{J^0}^\infty, \mathbb{Q})$, and $\mathbf{e}_k := \mathbf{1} \in H^{mid}(\mathbb{C}_{J^k}^N, W_{J^k}^\infty, \mathbb{Q})$.

The complex dimension of $\mathcal{H}_{W,G}$ is $pq + 1 - q$.

Denote by $\{r\}$ the fractional part of the real number r . We have

$$\Theta_x^{J^k} = \left\{ \frac{k}{p} \right\}, \quad \Theta_y^{J^k} = \left\{ \frac{k(p-1)}{pq} \right\}$$

and the transition number

$$\iota_{J^k} = \Theta_x^{J^k} - q_x + \Theta_y^{J^k} - q_y = \left\{ \frac{k}{p} \right\} + \left\{ \frac{k(p-1)}{pq} \right\} + \frac{1-p-q}{pq}$$

. For any $\alpha \in \mathcal{H}_{J^k}$, using the degree formula $\deg_{\mathbb{C}}(\alpha) = \deg_W(\alpha)/2 = \deg(\alpha)/2 + \iota_{J^k}$ we obtain

$$\begin{aligned} \deg_{\mathbb{C}}(y^{q-1} \mathbf{e}_0) &= 1 - \frac{(1-p-q)}{pq} = \frac{(p-1)(q-1)}{pq} = \hat{c}_W/2 \\ \deg_{\mathbb{C}} \mathbf{e}_k &= \left\{ \frac{k}{p} \right\} + \left\{ \frac{k(p-1)}{pq} \right\} + \frac{1-p-q}{pq}. \end{aligned}$$

3.2. Computation of the 3-correlators of genus 0. For convenience, we will write $y^{q-1} \mathbf{e}_0$ as \mathbf{e}_0 if there is no confusion, and define the set $\hat{\Lambda} := \Lambda \cup \{0\}$.

The computation of the genus zero, three point correlators $\langle a\mathbf{e}_i, b\mathbf{e}_j, c\mathbf{e}_k \rangle_0^W$ can be divided into four cases.

Case 1: $i = j = k = 0$. By dimension formula, we have

$$\langle y^{q-1} \mathbf{e}_0, y^{q-1} \mathbf{e}_0, y^{q-1} \mathbf{e}_0 \rangle_0^W = 0$$

.

Case 2: only one of i, j, k not equal to 0. The only non-zero correlator is $\langle \mathbf{e}_1, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0 \rangle_0^W$. Its value is the residue pairing of the element $y^{q-1}\mathbf{e}_0$ with itself, which is $-\frac{1}{q}$. Hence, we have

$$\eta_{0,0} = \langle \mathbf{e}_1, y^{q-1}\mathbf{e}_0, y^{q-1}\mathbf{e}_0 \rangle_0^W = -\frac{1}{q}$$

, and $\eta^{0,0} = -q$.

Case 3: $ijk \neq 0$.

Lemma 3.1. *If $ijk \neq 0$, then $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W \neq 0$ if and only if $i + j + k$ equals to $pq + 1$ or $2pq + 1$. Furthermore, $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = 1$ if and only if the corresponding line bundles satisfy $\deg|\mathcal{L}_x| = \deg|\mathcal{L}_y| = -1$ and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -q$ if and only if $\deg|\mathcal{L}_x| = -2$ and $\deg|\mathcal{L}_y| = 0$.*

Proof. If $ijk \neq 0$, $\sum_{t \in \{i,j,k\}} \deg_{\mathbb{C}}(\mathbf{e}_t) = \sum_{t \in \{i,j,k\}} (\{\frac{t}{p}\} + \{\frac{t(p-1)}{pq}\}) + \frac{3(1-p-q)}{pq}$.

In this case, the degrees of two orbifold line bundles are

$$\begin{aligned} \deg|\mathcal{L}_x| &= \frac{1}{p} - \{\frac{i}{p}\} - \{\frac{j}{p}\} - \{\frac{k}{p}\} \\ \deg|\mathcal{L}_y| &= \frac{p-1}{pq} - \{\frac{i(p-1)}{pq}\} - \{\frac{j(p-1)}{pq}\} - \{\frac{k(p-1)}{pq}\} \end{aligned}$$

By the dimension counting, $\langle a\mathbf{e}_i, b\mathbf{e}_j, c\mathbf{e}_k \rangle_0^W$ will vanish unless $\sum_t \deg_{\mathbb{C}}(\mathbf{e}_t) = \hat{c}_W$. Hence there is

$$\deg|\mathcal{L}_x| + \deg|\mathcal{L}_y| = -2$$

Since the degree of the resolved line bundles are integers, this shows that

$$(i + j + k)(p - 1) \equiv p - 1 \pmod{pq}.$$

Since $3 \leq i + j + k \leq 3pq - 3$ and $(p - 1, pq) = 1$, we must have $i + j + k = pq + 1, \text{ or } 2pq + 1$. Therefore

$$\deg|\mathcal{L}_x| = \frac{1}{p} - \{\frac{i}{p}\} - \{\frac{j}{p}\} - \{\frac{k}{p}\} < 0.$$

Since $\deg|\mathcal{L}_x| + \deg|\mathcal{L}_y| = -2$, we have two possibilities:

- (1) $\deg|\mathcal{L}_x| = \deg|\mathcal{L}_y| = -1$. In this case, by the concave axiom, we have $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = 1$.
- (2) $\deg|\mathcal{L}_x| = -2$ and $\deg|\mathcal{L}_y| = 0$. In the same way, we have $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -q$.

□

Corollary 3.2. *The metric has the form*

$$\eta_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha + \beta = pq \\ -1/q, & \text{if } \alpha = \beta = 0, \end{cases}$$

Proof. It is obvious since we have the relation $\eta_{\alpha\beta} = \langle \mathbf{e}_1, \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_0^W$.

□

Remark 3.3. We also have the following conclusions:

- (1) For fixed i and j , there is at most one k such that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W \neq 0$.
- (2) If $2 \leq (i + j) \leq pq$, then there must have $k = pq + 1 - (i + j)$.
- (3) if $(pq + 2) \leq (i + j) \leq (2pq - 2)$, then $k = 2pq + 1 - (i + j)$.

Case4: if only one of i, j, k equals to 0.

Lemma 3.4. Suppose only $k = 0$ in $\{i, j, k\}$, then $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $i + j = pq + 1$ and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W \neq 0$, and furthermore $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W = \pm 1$.

Proof. Suppose only $k = 0$, then $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W$ will vanish unless $\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) + \deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \hat{c}_W$. Since $\deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \hat{c}_W/2$, then this is equivalent to

$$\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) = \frac{(p-1)(q-1)}{pq}$$

On the other hand, by the composition axiom, we have

$$\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = \sum_{\alpha, \beta} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\beta \rangle_0^W + (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0} \quad (18)$$

Denote $\mathcal{L}_{x,i,j,i,j}$ and $\mathcal{L}_{y,i,j,i,j}$ by the orbifold line bundles corresponding to the G-decorated graph (i,j,i,j) respectively, then

$$\begin{aligned} \deg|\mathcal{L}_{x,i,j,i,j}| &= 2q_x - 2\Theta_x^{j_i} - 2\Theta_x^{j_j} = \frac{2}{p} - 2\{\frac{i}{p}\} - 2\{\frac{j}{p}\} \\ \deg|\mathcal{L}_{y,i,j,i,j}| &= 2q_y - 2\Theta_y^{j_i} - 2\Theta_y^{j_j} = \frac{2p-2}{pq} - 2\{\frac{i(p-1)}{pq}\} - 2\{\frac{j(p-1)}{pq}\}. \end{aligned}$$

Now we consider three cases.

(1) . Both p and q are odd.

In this case, at least one of $\eta_{\alpha\beta}, \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W, \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\beta \rangle_0^W$ vanishes, thus $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W \neq 0$.

Since the number $\deg|\mathcal{L}_{x,i,j,i,j}|$ and $\deg|\mathcal{L}_{y,i,j,i,j}|$ are integers, the dimension formula gives that $\deg|\mathcal{L}_{x,i,j,i,j}| + \deg|\mathcal{L}_{y,i,j,i,j}| = -2$.

Since

$$\deg|\mathcal{L}_{x,i,j,i,j}| = \frac{2}{p} - 2\{\frac{i}{p}\} - 2\{\frac{j}{p}\} < 0$$

and

$$\deg|\mathcal{L}_{y,i,j,i,j}| = \frac{2p-2}{pq} - 2\{\frac{i(p-1)}{pq}\} - 2\{\frac{j(p-1)}{pq}\} < 1$$

we must have either

$$(\frac{2}{p} - 2\{\frac{i}{p}\} - 2\{\frac{j}{p}\}, \frac{2p-2}{pq} - 2\{\frac{i(p-1)}{pq}\} - 2\{\frac{j(p-1)}{pq}\}) = (-1, -1)$$

or

$$(\frac{2}{p} - 2\{\frac{i}{p}\} - 2\{\frac{j}{p}\}, \frac{2p-2}{pq} - 2\{\frac{i(p-1)}{pq}\} - 2\{\frac{j(p-1)}{pq}\}) = (-2, 0)$$

Because p is odd, only the second case is possible, thus by the index-zero axiom, we have $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = -q$. Moreover, the equality $\frac{2p-2}{pq} - 2\{\frac{i(p-1)}{pq}\} - 2\{\frac{j(p-1)}{pq}\} = 0$ implies $i + j = pq + 1$.

The inverse conclusion is easy to see.

Now it is easy to check that $\langle \mathbf{e}_i, \mathbf{e}_{pq+1-i}, y^{q-1}\mathbf{e}_0 \rangle_0^W = \pm 1$.

(2) . p is even and q is odd.

In this case, the first term on the right hand of (18) is not zero if and only if $\alpha = \beta = \frac{pq}{2}$ and $\deg|\mathcal{L}_{x,i,j,\alpha}| = \deg|\mathcal{L}_{y,i,j,\alpha}| = -1$. This implies that $\deg|\mathcal{L}_{x,i,j,i,j}| = \deg|\mathcal{L}_{y,i,j,i,j}| = -1$, i.e. $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = 1$. Thus $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W = 0$. Here $i + j \neq pq + 1$.

If the first term on the right hand vanishes, then we have the same conclusion by the same argument in (1).

(3) . q is even.

Then by the assumption $(p-1, q) = 1$, p is even too. Hence $\frac{pq}{2} \notin \Lambda$, which concludes that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W \neq 0$ if and only if $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1} \mathbf{e}_0 \rangle_0^W \neq 0$. So we have $2(p-1)(1-i-j) \equiv 0 \pmod{pq}$ since $\deg[\mathcal{L}_{y,i,j,i,j}]$ is an integer. Then $i+j = \frac{pq}{2} + 1, pq+1$ or $\frac{3pq}{2} + 1$.

Assuming that $i+j = \frac{pq}{2} + 1$ or $\frac{3pq}{2} + 1$, we will have

$$i+j \equiv 1 \pmod{p}$$

and

$$\frac{(i+j)(p-1)}{pq} \equiv \frac{p-1}{2} + \frac{p-1}{pq} \equiv \frac{1}{2} + \frac{p-1}{pq} \pmod{1}.$$

Then $\{\frac{i}{p}\} + \{\frac{j}{p}\} = \frac{p+1}{p}$ and $\{\frac{i(p-1)}{pq}\} + \{\frac{j(p-1)}{pq}\} = \frac{p-1}{pq} + \frac{1}{2}$ or $\frac{p-1}{pq} + \frac{3}{2}$.

So we have $\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) = \Sigma_{i,j}(\{\frac{i}{p}\} + \{\frac{i(p-1)}{pq}\}) + \frac{1-p-q}{pq} = \frac{(p-1)(q-1)}{pq} + 1/2$ or $\frac{(p-1)(q-1)}{pq} + 3/2$ which contradicts with the degree formula.

So only $i+j = pq+1$ is possible, then we can proceed as before to reach the conclusion. \square

3.3. Generators and isomorphism. Now, we will prove there exist two generators of the quantum ring defined by Fan-Jarvis-Ruan-Witten theory. We need some preparation to prove this fact.

Lemma 3.5. *There exists a unique pair of integers k and m in $\hat{\Lambda}$ satisfying the two conditions:*

- (1) $\deg_{\mathbb{C}} \mathbf{e}_k = \frac{(q-1)}{pq}, \deg_{\mathbb{C}} \mathbf{e}_m = \frac{1}{q};$
- (2)

$$(k-1)(p-1) \equiv -1 \pmod{pq}$$

and

$$(m-2)(p-1) \equiv 1 \pmod{pq}$$

Proof. : Assume first that $p > 2$. Because $(p-1, q) = 1$, it is easy to see that the congruence equation

$$(k-1)(p-1) \equiv -1 \pmod{pq}$$

has a unique solution k such that $1 \leq k \leq pq-1$ and $p \nmid k$, this k will satisfy $p \mid (k-2)$, so

$$\deg_{\mathbb{C}} \mathbf{e}_k = \{\frac{k}{p}\} + \{\frac{k(p-1)}{pq}\} + \frac{1-p-q}{pq} = \frac{2}{p} + \frac{p-2}{pq} - \frac{1}{p} - \frac{p-1}{pq} = \frac{(q-1)}{pq}.$$

In the same way, we can find \mathbf{e}_m . Actually, there exists unique M , $1 \leq M \leq (pq-1)$, such that

$$(p-1)M \equiv 1 \pmod{pq}.$$

Now we have $k = pq+1-M$ and set $m = M+2$.

If $p = 2$, it is easy to get $M = 1, k = 0, m = 3$ and $k \equiv pq+1-M$. \square

In the following part, we still set $M = m-2$, where m is the special integer in Lemma 3.5.

Lemma 3.6. *There is a bijective map f between the set $\Delta := \{(s, t) \in \mathbb{Z} \oplus \mathbb{Z} \mid 0 \leq s \leq p-2, 0 \leq t \leq q-1\}$ and the set $\Lambda = \{i \in \mathbb{Z} \mid 1 \leq i \leq pq-1, p \nmid i\}$.*

Proof. We define a map $f : \Delta \longrightarrow \Lambda$ as follows.

If there exists $i \in \Lambda$ such that $i \equiv 1 + s(k-1) + t(m-1) \pmod{pq}$, then define $f(s, t) = i$. Since $1 + s(k-1) + t(m-1) = 1 + s(pq - M) + t(M+1) = spq + (M+1)(t-s) + (s+1)$, $p \mid spq + (M+1)(t-s)$, so $p \nmid 1 + s(k-1) + t(m-1)$. This thows that f is well defined on Δ .

Moreover, if $f(s, t) = f(s', t')$, then $1 + s(k-1) + t(m-1) \equiv 1 + s'(k-1) + t'(m-1) \pmod{pq}$. We have $s - s' \equiv (M+1)(s' + t' - s - t) \pmod{pq}$. This implies $p \mid (s - s')$ and we must have $s' = s$ and $(t' - t)(M+1) \equiv 0 \pmod{pq}$. Thus we have $t' = t$ and the map is injective.

Finally, since the sets Δ and Λ have the same cardinality $(p-1)q$, The map f is bijective. \square

Now for each $i \in \Lambda$, we can identify $\mathbf{e}_i, \mathbf{e}_{1+s(k-1)+t(m-1)}$ and (s, t) , where $f(s, t) = i$.

Lemma 3.7. $(s, t) \star (u, v) = (s + u, t + v)$ if $0 \leq s + u \leq p - 2$ and $0 \leq t + v \leq q - 1$.

Proof. Let $i = f(s, t), j = f(u, v)$, then

$$(s, t) \star (u, v) = \mathbf{e}_i \star \mathbf{e}_j = \sum_{\alpha \in \beta} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta$$

Because $i + j \equiv 1 + s(k-1) + t(m-1) + 1 + u(k-1) + v(m-1) \equiv (M+1)(t+v-s-u) + (2+s+u) \pmod{pq}$, then $i + j \neq 1 \pmod{pq}$. Now by discussions above, we know there exists at most one element $\alpha \in \Lambda$ such that $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \neq 0$, and $\alpha \equiv (pq+1) - (1 + s(k-1) + t(m-1) + 1 + u(k-1) + v(m-1)) \equiv -(M+1)(t+v-s-u) - (1+s+u) \pmod{pq}$.

So we have $\deg[\mathcal{L}_x] = \frac{1}{p} - \{\frac{i}{p}\} - \{\frac{j}{p}\} - \{\frac{\alpha}{p}\} = \frac{1}{p} - \frac{1+s}{p} - \frac{1+u}{p} - \frac{p-1-s-u}{p} = -1$. Thus $\deg[\mathcal{L}_y] = -1$ also and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W = 1$.

Now $(s, t) \star (u, v) = \mathbf{e}_i \star \mathbf{e}_j = \mathbf{e}_{pq-\alpha} = \mathbf{e}_{1+(s+u)(k-1)+(t+v)(m-1)} = (s+u, t+v)$. \square

Lemma 3.8. $(p-2, 0) \star (1, 0) = \mp q \cdot y^{q-1} \mathbf{e}_0$

Proof. $(p-2, 0) \star (1, 0) = \mathbf{e}_{1+(p-2)(k-1)} \star \mathbf{e}_k = \sum_{\alpha \in \beta} \langle \mathbf{e}_{1+(p-2)(k-1)}, \mathbf{e}_k, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta$. Since $\mathbf{e}_{1+(p-2)(k-1)} = \mathbf{e}_M$ and $M + k = pq + 1$, we have $\langle \mathbf{e}_{1+(p-2)(k-1)}, \mathbf{e}_k, \mathbf{e}_\alpha \rangle_0^W \neq 0$ if and only if $\alpha = 0$.

Now we have $(p-2, 0) \star (1, 0) = \langle \mathbf{e}_M, \mathbf{e}_k, y^{q-1} \mathbf{e}_0 \rangle_0^W \eta^{0,0} y^{q-1} \mathbf{e}_0 = \mp q \cdot y^{q-1} \mathbf{e}_0$. \square

Define $(p-1, 0) = \mp q \cdot y^{q-1} \mathbf{e}_0$. Then we have the representation $(p-2, 0) \star (1, 0) = (p-1, 0)$ and $(p-1-s, 0) \star (s, 0) = (p-1, 0)$ for any $0 \leq s \leq p-2$.

Lemma 3.9. $(p-1, 0) \star (0, 1) = 0$.

Proof. $(p-1, 0) \star (0, 1) = \mp q \cdot y^{q-1} \mathbf{e}_0 \star \mathbf{e}_m = \mp q \sum_{\alpha \in \beta} \langle y^{q-1} \mathbf{e}_0, \mathbf{e}_m, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta$.

Now $\langle y^{q-1} \mathbf{e}_0, \mathbf{e}_m, \mathbf{e}_\alpha \rangle_0^W \neq 0$ only if $m + \alpha = pq + 1$. This implies $\alpha = pq + 1 - m = pq - (M+1)$ and $p \mid \alpha$ which contradicts with $\alpha \in \Lambda$.

Thus $\langle y^{q-1} \mathbf{e}_0, \mathbf{e}_m, \mathbf{e}_\alpha \rangle_0^W = 0$ for each $\alpha \in \Lambda$ and $y^{q-1} \mathbf{e}_0 \star \mathbf{e}_m = 0$. \square

Define $(p, 0) = (1, 0) \star (p-1, 0)$.

Lemma 3.10. $(p, 0) + q(0, q-1) = 0$.

Proof. $(p, 0) = (1, 0) \star (p-1, 0) = \mathbf{e}_k \star (\mp q \cdot y^{q-1} \mathbf{e}_0) = \mp q \sum_{\alpha \in \beta} \langle \mathbf{e}_k, y^{q-1} \mathbf{e}_0, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \mathbf{e}_\beta = \mp q \langle \mathbf{e}_k, y^{q-1} \mathbf{e}_0, \mathbf{e}_M \rangle_0^W \eta^{M, pq-M} \mathbf{e}_{pq-M} = -q \mathbf{e}_{pq-M} = -q(0, q-1)$. \square

Lemma 3.11. $(s, t) \star (u, v) = 0$ if $t + v \geq q$.

Proof.

$$\begin{aligned}
& (s, t) \star (u, v) \\
&= (s, 0) \star [(0, t) \star (0, v)] \star (u, 0) \\
&= (s, 0) \star [(0, t) \star (0, q-1-t) \star (0, 1) \star (0, v+t-q)] \star (u, 0) \\
&= (s, 0) \star [(0, q-1) \star (0, 1)] \star (0, v+t-q) \star (u, 0) \\
&= (s, 0) \star \left[-\frac{1}{q} (p, 0) \star (0, 1) \right] \star (0, v+t-q) \star (u, 0) \\
&= (s, 0) \star 0 \star (0, v+t-q) \star (u, 0) \\
&= 0.
\end{aligned}$$

□

Lemma 3.12. $(s, t) \star (u, v) = 0$ if $s + u \geq p - 1$ and $t + v \neq 0$.

Proof.

$$\begin{aligned}
& (s, t) \star (u, v) \\
&= (0, t) \star [(s, 0) \star (p-1-s, 0)] \star (s+u+1-p, v) \\
&= (0, t) \star (p-1, 0) \star (s+u+1-p, v) \\
&= 0.
\end{aligned}$$

□

Lemma 3.13. $(s, 0) \star (u, 0) = -q(s+u-p, q-1)$ if $0 \leq s, u \leq p-1, p \leq s+u \leq 2p-2$.

Proof. $(s, 0) \star (u, 0) = (s, 0) \star (p-s, 0) \star (s+u-p, 0) = (p, 0) \star (s+u-p, 0) = -q(0, q-1) \star (s+u-p, 0) = -q(s+u-p, q-1)$ □

Now by the above lemmas, we obtain the following theorem.

Theorem 3.14. The two generators \mathbf{e}_k and \mathbf{e}_m in Lemma 3.5 generate the quantum ring of $X^p + XY^q$, where $(p-1, q) = 1$. The multiplication is given by

- (1) $\mp q \cdot y^{q-1} \mathbf{e}_0 = \mathbf{e}_k^{p-1}$;
- (2) $\mathbf{e}_i = \mathbf{e}_k^s \star \mathbf{e}_m^t$, if $i \in \Lambda$ such that $f(s, t) = i$ for $(s, t) \in \Delta$.

Moreover, we have two relations $\mathbf{e}_k^{p-1} \star \mathbf{e}_m = (p-1, 0) \star (0, 1) = 0$ and $\mathbf{e}_k^p + q\mathbf{e}_m^{q-1} = (p, 0) + q(0, q-1) = 0$.

This theorem demonstrate the phenomenon of mirror symmetry between two dual singularities:

Corollary 3.15. If $(p-1, q) = 1$, then $\mathcal{H}_{W,G} \cong \mathcal{L}_{\check{W}}$, where $\check{W} = X^p Y + Y^q$ is the dual singularity.

Proof. Define the \mathbb{C} -algebra epimorphism $F : \mathbb{C}[X, Y] \longrightarrow \mathcal{H}_{W,G}$ such that $F(X) = \mathbf{e}_k$ and $F(Y) = \mathbf{e}_m$. Then $F(X^{p-1}Y) = \mathbf{e}_k^{p-1} \star \mathbf{e}_m = 0$ and $F(X^p + qY^{q-1}) = \mathbf{e}_k^p + q\mathbf{e}_m^{q-1} = 0$. Thus $X^{p-1}Y, X^p + qY^{q-1} \in \text{Ker}(F)$, we have a \mathbb{C} -algebra epimorphism $\bar{F} : \mathbb{C}[X, Y]/(pX^{p-1}Y, X^p + qY^{q-1}) \longrightarrow \mathcal{H}_{W,G}$. $\mathbb{C}[X, Y]/(pX^{p-1}Y, X^p + qY^{q-1})$ is just the Milnor ring of the singularity $\check{W} = X^p Y + Y^q$. We have $\dim_{\mathbb{C}} \mathbb{C}[X, Y]/(pX^{p-1}Y, X^p + qY^{q-1}) = \dim_{\mathbb{C}} \mathcal{H}_{W,G} = pq - q + 1$. Those facts shows that \bar{F} is a \mathbb{C} -algebra isomorphism. □

4. QUANTUM RING OF $X^p + XY^q$; $(p-1, q) = d > 1$

4.1. **Basic calculation.** We have the same fractional degrees and the central charge:

$$q_x = \frac{1}{p}, \quad q_y = \frac{p-1}{pq}, \quad \hat{c}_W = \frac{2(p-1)(q-1)}{pq}$$

Let $\xi = \exp(\frac{2\pi i}{pq})$, and λ acts on $\mathcal{D}_W \omega$ by (ξ^{-q}, ξ) . Then λ generates the maximal admissible abelian group $G = \mathbb{Z}/(pq)\mathbb{Z}$.

Now $\Theta_x^J = \frac{p-1}{p}$, and $\Theta_y^J = \frac{1}{pq}$.

The G-invariant state space of the polynomial W is also:

$\mathcal{H}_{W,G} = \langle y^{q-1} \mathbf{e}_0, \mathbf{e}_k | k \in \Lambda \rangle$, where $\mathbf{e}_0, \mathbf{e}_k, \Lambda$ are defined as before.

The dimension is $\dim_{\mathbb{C}} \mathcal{H}_{W,G} = pq + 1 - q$. We have the degree computation:

$$\begin{aligned} \Theta_x^{J^k} &= \left\{ \frac{p-k}{p} \right\}, \quad \Theta_y^{J^k} = \left\{ \frac{k}{pq} \right\} \\ \iota_{J^k} &= \Theta_x^{J^k} - q_x + \Theta_y^{J^k} - q_y = \left\{ \frac{p-k}{p} \right\} + \left\{ \frac{k}{pq} \right\} - \frac{p+q-1}{pq} \\ \deg_{\mathbb{C}}(y^{q-1} \mathbf{e}_0) &= 1 - \frac{(p+q-1)}{pq} = \frac{(p-1)(q-1)}{pq} = \hat{c}_W/2 \\ \deg_{\mathbb{C}} \mathbf{e}_k &= \left\{ \frac{p-k}{p} \right\} + \left\{ \frac{k}{pq} \right\} + \frac{1-p-q}{pq}. \end{aligned}$$

Remark 4.1. Since $\deg \mathbf{e}_{p-1} = 0$ in this case, So \mathbf{e}_{p-1} will become the unit in the quantum ring.

4.2. **Computation of the 3 point correlators in genus 0.** The computation of the genus zero three point correlators $\langle a\mathbf{e}_i, b\mathbf{e}_j, c\mathbf{e}_k \rangle_0^W$ is also divided into four cases:

Case 1: $i = j = k = 0$. By dimension formula, we have

$$\langle y^{q-1} \mathbf{e}_0, y^{q-1} \mathbf{e}_0, y^{q-1} \mathbf{e}_0 \rangle_0^W = 0.$$

Case 2: $j = k = 0, i \in \Lambda$. The only non-zero correlator is $\langle \mathbf{e}_{p-1}, y^{q-1} \mathbf{e}_0, y^{q-1} \mathbf{e}_0 \rangle_0^W$ and $\langle \mathbf{e}_{p-1}, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -\frac{1}{q}$.

Case 3: $i, j, k \in \Lambda$.

Lemma 4.2. If $i, j, k \in \Lambda$, then there are two cases such that $\langle a\mathbf{e}_i, b\mathbf{e}_j, c\mathbf{e}_k \rangle_0^W \neq 0$.

- (1) $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -q$ if and only if $i + j + k = p - 1$
- (2) $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = 1$ if and only if $i + j + k = pq + p - 1$

Proof. We have the computation:

$$\begin{aligned} \Sigma_{i,j,k} \deg_{\mathbb{C}}(\mathbf{e}_i) &= \Sigma_{i,j,k} \left(\left\{ \frac{p-i}{p} \right\} + \left\{ \frac{i}{pq} \right\} \right) + \frac{3(1-p-q)}{pq} \\ \deg |\mathcal{L}_x| &= \frac{1}{p} - \left\{ \frac{p-i}{p} \right\} - \left\{ \frac{p-j}{p} \right\} - \left\{ \frac{p-k}{p} \right\} \\ \deg |\mathcal{L}_y| &= \frac{p-1}{pq} - \left\{ \frac{i}{pq} \right\} - \left\{ \frac{j}{pq} \right\} - \left\{ \frac{k}{pq} \right\} \end{aligned}$$

If $\langle \mathbf{a}\mathbf{e}_i, \mathbf{b}\mathbf{e}_j, \mathbf{c}\mathbf{e}_k \rangle_0^W \neq 0$, then by dimension formula, there is

$$i + j + k \equiv p - 1 \pmod{pq}.$$

Thus $i + j + k = p - 1, pq + p - 1$ or $2pq + p - 1$. But the later case implies that $\Sigma_i \{ \frac{p-i}{p} \} = \frac{1}{p}$, which is impossible.

If $i + j + k = p - 1$, then $(\deg|\mathcal{L}_x|, \deg|\mathcal{L}_y|) = (-2, 0)$ and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = -q$.

If $i + j + k = pq + p - 1$, then $(\deg|\mathcal{L}_x|, \deg|\mathcal{L}_y|) = (-1, -1)$ and $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \rangle_0^W = 1$. \square

Remark 4.3. The metric $\eta_{\alpha\beta}$ has the same form as in Corollary 3.2.

Case 4: $k = 0, i, j \in \Lambda$.

Lemma 4.4. $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $i + j = p - 1$.

Moreover $\langle \mathbf{e}_i, \mathbf{e}_{p-1-i}, y^{q-1}\mathbf{e}_0 \rangle_0^W = \pm 1$.

Proof. By the cutting formula, we have

$$\begin{aligned} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W &= \sum_{\alpha, \beta \in \Lambda} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\alpha \rangle_0^W \eta^{\alpha\beta} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_\beta \rangle_0^W + (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0} \\ &= \sum_{l \in \Lambda} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_l \rangle_0^W \eta^{l, pq-l} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq-l} \rangle_0^W + (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0} \end{aligned}$$

Now

$$\begin{aligned} \deg|\mathcal{L}_{x,i,j,i,j}| &= \frac{2}{p} - 2\{\frac{p-i}{p}\} - 2\{\frac{p-j}{p}\} \\ \deg|\mathcal{L}_{y,i,j,i,j}| &= \frac{2(p-1)}{pq} - 2\{\frac{i}{pq}\} - 2\{\frac{j}{pq}\} \end{aligned}$$

and the degree formula $\deg_{\mathbb{C}}(\mathbf{e}_i) + \deg_{\mathbb{C}}(\mathbf{e}_j) + \deg_{\mathbb{C}}(y^{q-1}\mathbf{e}_0) = \hat{c}_W$ shows that $\deg|\mathcal{L}_{x,i,j,i,j}| + \deg|\mathcal{L}_{y,i,j,i,j}| = -2$.

Since $-3 < \deg|\mathcal{L}_{x,i,j,i,j}| < 0$. Then $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W \neq 0$ if and only if $(\deg|\mathcal{L}_{x,i,j,i,j}|, \deg|\mathcal{L}_{y,i,j,i,j}|) = (-2, 0)$ or $(-1, -1)$.

On the other hand, $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$ or -1 implies $i + j = p - 1$ or $p - 1 + \frac{pq}{2}$ (if $2 \mid (pq)$).

Then we have three cases:

(i) If q is even, $\frac{pq}{2}$ is not belong to Λ , then $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = (\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0}$.

Thus $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ implies $i + j = p - 1$ or $\frac{pq}{2} + p - 1$.

If $i + j = p - 1$, then $\deg|\mathcal{L}_{x,i,j,i,j}| = \frac{2}{p} - 2\{\frac{p-i}{p}\} - 2\{\frac{p-j}{p}\} = -2$ and $\deg|\mathcal{L}_{y,i,j,i,j}| = 0$, $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = -q$. If $i + j = \frac{pq}{2} + p - 1$, then $i + j \equiv -1 \pmod{p}$. So $\deg|\mathcal{L}_{x,i,j,i,j}| = \frac{2}{p} - 2\{\frac{p-i}{p}\} - 2\{\frac{p-j}{p}\} = -2$ and $\deg|\mathcal{L}_{y,i,j,i,j}| = -1$, which contradict with the fact that $\deg|\mathcal{L}_{x,i,j,i,j}| + \deg|\mathcal{L}_{y,i,j,i,j}| = -2$;

Therefore $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $i + j = p - 1$.

(ii) If p is even and q is odd, then $\frac{pq}{2} \in \Lambda$.

If $i + j = \frac{pq}{2} + p - 1$, $\langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = 1$, then we have

$$\sum_{l \in \Lambda} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_l \rangle_0^W \eta^{l, pq-l} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{pq-l} \rangle_0^W = \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{\frac{pq}{2}} \rangle_0^W \eta^{\frac{pq}{2}, \frac{pq}{2}} \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{\frac{pq}{2}} \rangle_0^W = 1$$

Thus $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W = 0$.

If $i + j = p - 1$, we have

$$(\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0} = \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W = -q.$$

Thus $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1}\mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $i + j = p - 1$.

- (iii) If both p, q are odd, then $\frac{pq}{2}$ is not an integer. Hence $(\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1} \mathbf{e}_0 \rangle_0^W)^2 \eta^{0,0} = \langle \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j \rangle_0^W$. So $\langle \mathbf{e}_i, \mathbf{e}_j, y^{q-1} \mathbf{e}_0 \rangle_0^W \neq 0$ if and only if $i + j = p - 1$.

□

4.3. Generators and isomorphism.

Lemma 4.5. *There is a bijective map g between the set $\Delta = \{(s, t) \in \mathbb{Z} \oplus \mathbb{Z} \mid 0 \leq s \leq p-2, 0 \leq t \leq q-1\}$ and the set $\Lambda = \{i \in \mathbb{Z} \mid 1 \leq i \leq pq-1, p \nmid i\}$.*

Proof. We define a map $g : \Delta \longrightarrow \Lambda$ as follows:

If there exists $i \in \Lambda$ such that $i \equiv p-1-s+tp \pmod{pq}$, then $g(s, t) = i$. Since $p \nmid tp-1-s$, this map is well defined on the whole set Δ .

Moreover, It is easy to verify that $g(s, t) = g(s', t')$ if and only if $s' = s$ and $t' = t$. So the map is injective.

Finally, the cardinalities of the set Δ and the set Λ are both equal to $(p-1)q$, we conclude that the map g is bijective. □

Now for any $i \in \Lambda$, we identify \mathbf{e}_i and (s, t) as the same element if $g(s, t) = i$. Then $(1, 0) = \mathbf{e}_{p-2}$ and $(0, 1) = \mathbf{e}_{2p-1}$. If we replace \mathbf{e}_k and \mathbf{e}_m in Theorem 3.14 by \mathbf{e}_{p-2} and \mathbf{e}_{2p-1} respectively, then it is straightforward as in Section 3 to prove the following theorem.

Theorem 4.6. *The two elements \mathbf{e}_{p-2} and \mathbf{e}_{2p-1} generate the quantum ring of quasi-homogeneous polynomial $X^p + XY^q$ for $(p-1, q) = d > 1$. The multiplication is determined by the following relations.*

- \mathbf{e}_{p-1} is the unit of this ring;
- $\mp q \cdot y^{q-1} \mathbf{e}_0 = \mathbf{e}_{p-2}^{p-1}$;
- $\mathbf{e}_i = \mathbf{e}_{p-2}^s \star \mathbf{e}_{2p-1}^t$ for each $i \in \Lambda$, where $(s, t) \in \Delta$ such that $g(s, t) = i$,

and

- $\mathbf{e}_{p-2}^{p-1} \star \mathbf{e}_{2p-1} = (p-1, 0) \star (0, 1) = 0$;
- $\mathbf{e}_{p-2}^p + q \mathbf{e}_{2p-1}^{q-1} = (p, 0) + q(0, q-1) = 0$.

Corollary 4.7. *If $(p-1, q) = d > 1$, and $G = \langle \lambda \rangle$, then $\mathcal{H}_{W,G} \cong \mathcal{L}_{\check{W}}$. where $\check{W} = X^p Y + Y^q$ is the dual singularity.*

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